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Relating Position Uncertainty to Maximum Conjunction Probability

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Abstract

The effects of positional uncertainty on the Gaussian probability computation for orbit conjunction are examined and an upper bound determined. Relative motion between two objects is assumed linear for a given encounter with time-invariant position covariance. A method is developed to assess the maximum probability for various satellite sizes, encounter geometries, and covariance sizes and shapes. The associated standard deviation then defines the boundary of probability dilution. The assertion is made that orbit positions should be sufficiently accurate to avoid this dilution region. This work shows how to calculate the upper bounds of probability by assuming worst-case covariance orientation and size. Power series approximations are developed for aspect ratios ranging from 1 to 50 to capture 99% of all conjunction possibilities. An analytical approximation is also given for an infinite aspect ratio to capture all possibilities. These expressions can be used as a simple pre-filter or to determine worst-case scenarios. Although desired, the actual covariances are not needed. What is needed is the ratio of major-to-minor axes of the projected combined covariance ellipse, the object sizes, and the relative distance at the point of closest approach.

Introduction

Probability calculations for conjunction analysis should ensure sufficient accuracy to give meaningful results. Because all operational decisions are ultimately made with respect to the amount of acceptable risk, the action threshold should not be based on an unacceptable miss distance but rather on an unacceptable collision probability. This is already done with the International Space Station and Space Transportation System where avoidance maneuvers are initiated if the collision probability exceeds an acceptable risk threshold. If the positional uncertainty is very large, a Gaussian calculation will produce a low conjunction probability. Although mathematically correct, the resulting probability may give a false sense of confidence that a conjunction is not likely to occur. Such a low probability may, in fact, indicate that the data are not of sufficient accuracy to produce an operationally meaningful result.

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The accuracy of positional covariance matrices resulting from Least Squares Orbit Determination of sparse data is questionable. Covariance matrices formed in this manner often provide overly optimistic results. Frisbee and Foster [1] noted that "The primary problem with state error covariances determined from observations of objects in Earth orbit is that they are not truly reflective of the uncertainties in the dynamic environment." To address this concern, they devised a method to scale covariances provided to NASA by Air Force Space Command. This paper details a way to address the inaccuracy of covariance matrices by determining a mathematical upper bound that will not be exceeded regardless of satellite positional uncertainty.

Much work has been done to address the computing of collision probability for neighboring space objects [1–9] and some work has been done to examine accuracy requirements associated with those computations [10]. Typically, one determines if and when a secondary object will transgress a user-defined safety zone. The uncertainties associated with position are represented by three-dimensional Gaussian probability densities. These densities take the form of covariance matrices and can be obtained from the owner-operators or independent surveillance sources such as the US Satellite Catalog (Special Perturbations). Positions and covariances are propagated to the time of closest approach.

It is possible to find the absolute worst-possible covariance size and orientation that maximizes the probability for a given encounter where the object sizes are known and the closest-approach distance is fixed. It is also possible to find covariance parameters that maximize the probability for various covariance shapes as determined by the aspect ratio (i.e., the ratio of major-to-minor axes of the projected combined covariance ellipse). Charts were previously created to assess such probabilities [11]; this work details the mathematics used to create those charts.

If the maximum probability is below a predefined user threshold, then no further calculations are needed. Even in the absence of known covariances, the numerical approximations that follow will still provide the user with worst-case collision potential for aspect ratios ranging from 1 to 50, capturing 99% of all conjunction possibilities. Beyond those bounds, an iterative search method is recommended. Such analysis can be insightful when one only has knowledge of the miss distance and physical object sizes.

Collision Probability Computation

There are many assumptions that reduce the problem's complexity. The physical objects are treated as spheres, thus eliminating the need for attitude information (Fig. 1). Their relative motion is considered linear for the encounter by assuming the effect of relative acceleration is dwarfed by that of the velocity. The positional errors are assumed to be zero-mean, Gaussian, uncorrelated, and constant for the encounter. The relative velocity at the point of closest approach is deemed sufficiently large to ensure a brief encounter time and static covariance. The encounter region is defined when one object is within a standard deviation (σ) combined covariance ellipsoid shell scaled by a factor of n. This user-defined, three-dimensional, n-σ shell is centered on the primary object; n is typically in the range of three to eight to accommodate conjunction possibilities ranging from 97.070911% to 99.999999%.

Because the covariances are expected to be uncorrelated, they are simply summed to form one, large, combined, covariance ellipsoid that is centered at the primary object (Fig. 2). The secondary object passes quickly through this ellipsoid

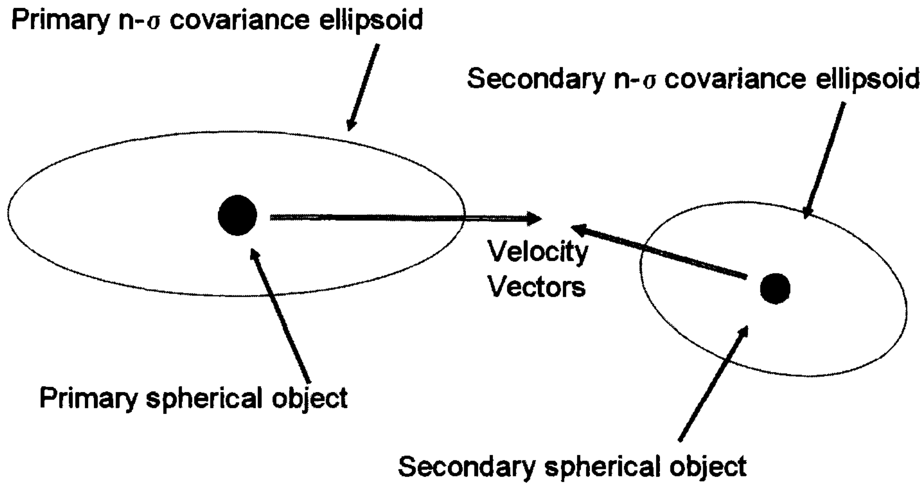

FIG. 1. Conjunction Encounter Geometry.

creating a tube-shaped path. A conjunction occurs if the secondary sphere touches the primary sphere, i.e., when the distance between the two projected object centers is less than the sum of their radii. The radius of this collision tube is enlarged to accommodate all possibilities of the secondary touching the primary by combining the radii of both objects.

A plane perpendicular to the relative velocity vector is formed and the combined object and covariance ellipsoid are projected onto this encounter plane. As stated previously, the encounter region is defined by an n-σ shell determined by the user to sufficiently account for conjunction possibilities. Within that shell the tube is

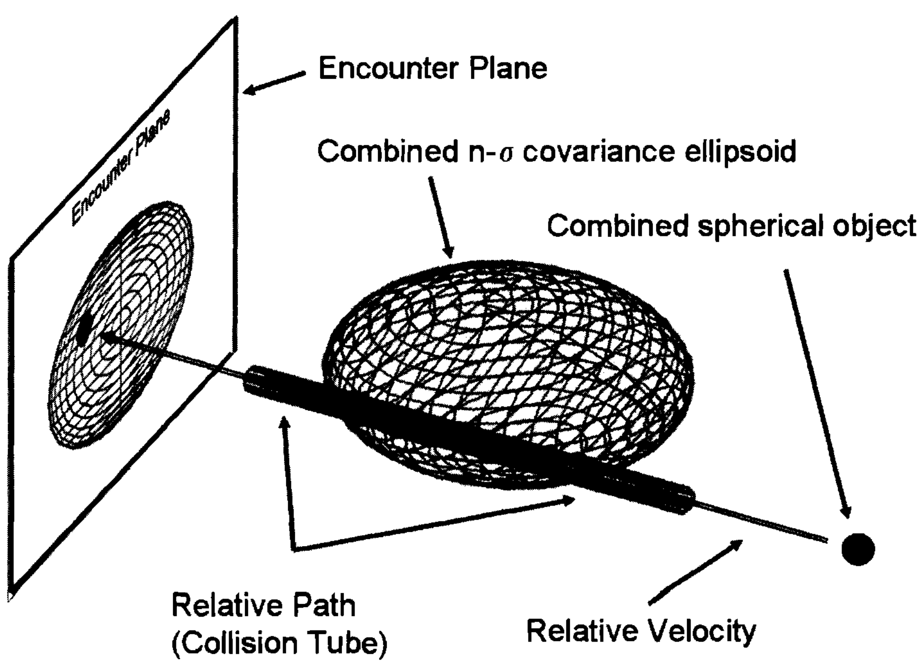

FIG. 2. Conjunction Encounter Visualization and Reduction.

straight and rapidly traversed, allowing a decoupling of the dimension associated with the tube path (i.e., relative velocity). The tube becomes a circle on the projected encounter plane. Likewise, the covariance ellipsoid becomes an ellipse (Fig. 3).

The relative velocity vector (decoupled dimension) is associated with the time of closest approach. The conjunction assessment here is concerned with cumulative probability over the time it takes to span the n-σ shell, not an instantaneous probability at a specific time within the shell. Along this dimension, integration of the probability density across the shell produces a number very near unity, meaning the close approach will occur at some time within the shell with near absolute certainty. Thus the cumulative collision probability is reduced to a two-dimensional problem in the encounter plane that is then multiplied by the decoupled dimension's probability. By rounding the latter probability to one, it is eliminated from further calculations.

The resulting two-dimensional probability equation in the encounter plane is given as

$$P = \frac{1}{2\pi\sigma x \sigma y}\int_{-OBJ}^{OBJ}\int_{-\sqrt{OBJ^2-(x)^2}}^{\sqrt{OBJ^2-(x)^2}} \exp\left[\left(\frac{-1}{2}\right)\left[\left(\frac{x+xm}{\sigma x}\right)^2 + \left(\frac{y+ym}{\sigma y}\right)^2\right]\right] \mathrm{d}y\,\mathrm{d}x \tag{1}$$

where OBJ is the combined object radius, x lies along the minor axis, y lies along the major axis, xm and ym are the respective components of the projected miss distance, and σx and σy are the corresponding standard deviations. For the formulation that follows, the aspect ratio AR is incorporated as a multiple of the minor axis standard deviation (AR $\geq$ 1) and equation (1) is expressed as

$$P = \frac{1}{2\pi\sigma x^2 AR}\int_{-OBJ}^{OBJ}\int_{-\sqrt{OBJ^2-(x)^2}}^{\sqrt{OBJ^2-(x)^2}} \exp\left[\left(\frac{-1}{2}\right)\left[\left(\frac{x+xm}{\sigma x}\right)^2 + \left(\frac{y+ym}{\sigma x AR}\right)^2\right]\right] \mathrm{d}y\,\mathrm{d}x \tag{2}$$

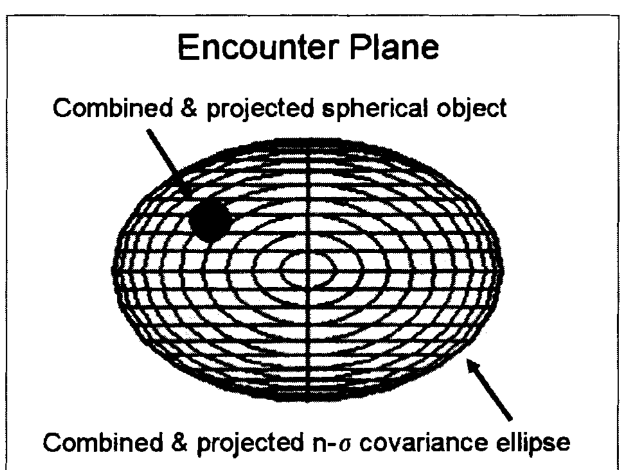

FIG. 3. Projection onto the Encounter Plane.